

The Breath at the Wall: A Balanced, Steady Window of Eight Postings Is One Act

Jonathan Washburn

Recognition Physics Research Institute, Austin, Texas
ORCID 0009-0001-8868-7497

jon@recognitionphysics.org

Abstract

The premises of Recognition Science (RS) never pick the next posting of the ledger: from every balanced journal several next postings are admissible, completable and equally cheap, and equally cheap candidates differ only by a renaming of sites and an orientation, a ratio against its reciprocal. We show that two conditions on a window of eight postings close that gap. Balance (four postings of each sign) and steady build-up (each sign forms a single run) admit exactly two windows, and the two differ only in which way round they run. On four accounts with equal units the window is one act, which we call the breath: it debits the four accounts in turn and then credits them in the same order. Its running totals are the tones 0 1 2 3 4 3 2 1 0; each posting changes the account that has held its state longest; its period $2(D+1)$ equals the 2^D corners of the D-cube only when $D = 3$; and read through three bits it walks every corner of the cube. Then, on RS's own order of four accounts posted in turn, one premise forces the two conditions: the ledger forms the breath in every window exactly when no account ever holds more than one unit and a fresh opening is a plus. Equivalently, each posting takes the side of least cost on the account the order names; that is RS's own side rule, with the cost read on balances, and the side it leaves open at an empty account is the orientation. The premise returns every unit four postings later even where nothing completes, and it needs exactly four accounts in turn. It picks the side, not the account: the breath is not least cost among all postings. What stays open is whether the ledger's accounts hold at most one unit, which RS has at model status, and what one tick of the ledger is, including the order. The two conditions came from Walter Russell's *The Universal One* (1926), which we read in full, tested against measurement and matched to RS principle by principle; Section 9 tells that history.

1 Introduction

1.1 The wall

Reality is a ledger of recognition events. Each event compares two sites with a positive ratio x, and its cost is

$$J(x) = \frac{x + x^{-1}}{2} - 1 = \frac{(x-1)^2}{2x}. \tag{1}$$

Among continuous costs it is the only one that satisfies the composition law of comparison, normalized and calibrated [3]. It is never negative, it vanishes only at $x = 1$, and $J(x) = J(1/x)$. Every event has a reciprocal event with the inverse ratio, so the ledger is double entry, and a closed ledger is balanced. The structural axioms of a conserved comparison ledger fix its scale, its cost and its ratio [4]. Space has three dimensions, and three binary coordinates give the eight states of the eight-tick cycle (proved in work not yet published).

What the ledger's premises do not say is what happens next. Section 6 states this exactly. From every balanced journal, two different next postings are both admissible, both completable and of equal cost. Equally cheap candidates differ only by a renaming of the sites and by the orientation: a ratio r against $1/r$. A static cost and its strict minimum cannot select a motion: the cheapest posting is rest. And no single field carries the action of every posting at once; each act has its own. We call this the wall. Behind it stands a second question that the premises also leave open: what one tick of the ledger is.

1.2 Where the answer came from

Walter Russell's *The Universal One* (1926) builds a whole account of nature on one substance, one force and a pulse of two opposed impulses [1]. He describes that pulse as a breath. "Each octave is a separate and complete sequence of inhalation and exhalation" [1, p. 88]. His octave chart puts its arithmetic in three lines: "Two opposed tones = one unit. Four units = one octave. Ten octaves = one cycle" [1, octave chart following p. 40]. He says the four units are equal: "The energy of each of the four units is exactly equal to that of each of the others" [1, law 125, p. 162]. He requires balance, "Every plus pressure total must be balanced by a minus one to maintain a system in equilibrium" [1, law 123, p. 160], and an order in time: "each opposite is cumulatively preponderant in sequence" [1, law 26, p. 67].

We read Russell as a source of structure, with logic and measurement deciding. We recorded every claim in the 1926 book, 4,047 distinct claims in all, and tested the 467 that measurement can decide. His numbers fail where his own laws go beyond the measurements of his day: 216 of the 467 disagree with measurement. His organizing structure holds up far better. Eleven of his sixteen first principles are RS's own. Where he says more than RS, the subject is always the same: what moves. That is exactly where RS has its wall. Section 9 tells this history in full.

1.3 The result

Take a window of eight postings, each a unit opened (+) or closed (−). Call it balanced when it has four of each sign, and steady when each sign forms a single run, so that one opposite builds up and then the other. These are Russell's balance and his cumulative preponderance in sequence, made exact.

- **The window is unique up to orientation** (Theorem 4). Exactly two windows are balanced and steady, ++++−−−− and −−−−++++, and each is the other with every sign reversed. Of the seventy balanced windows, steadiness keeps two.
- **On four accounts it is one act** (Theorem 8). With equal units every loan stays open exactly half the window. The act debits four accounts in turn and then credits them in the same order. Its tones are 0 1 2 3 4 3 2 1 0, each posting changes the account that has held its state longest, and its eight states are distinct.
- **Three dimensions** (Theorems 10 and 12). On $D + 1$ accounts the act has period $2(D + 1)$, and the D-cube has 2^D corners. They agree only at $D = 3$. Read through three bits, with the two opposite accounts as one bit, the act walks every corner of the cube, one bit per posting.
- **At the wall** (Section 6). As a law of content the breath selects the next posting where the premises select nothing, it permits loans, and opening then closing first in, first out returns the journal to balance. On the postings, the only freedom it leaves is a renaming of the accounts and the orientation, which are the freedoms the wall's own symmetries leave.
- **One premise yields it** (Theorem 26). Post four accounts in turn, which is RS's own worked example of a legal clock. Then every window is the breath exactly when no account ever holds more than one unit and a fresh opening is a plus. Equivalently, each posting takes the side of least cost on the account the order names, which is RS's own side rule with the cost read on balances (Propositions 23 and 25); RS leaves the side at an empty account open, and that is the orientation. The premise returns each unit four postings later whether or not anything completes, and needs exactly four accounts in turn (Corollaries 29 and 30). It picks the side and not the account (Proposition 27), and it rests on two features: the cost reads a signed balance, and every tick posts (Proposition 28).

What stays open is whether the ledger's accounts hold at most one unit, that is, whether the side rule's reading of balances is the ledger's own, which RS has at model status; and what one tick of the ledger is, including whether it posts four accounts in turn. Section 7 gives what is known about clocks and registers on the four-account record, as conditional results, and Section 8 states the open questions as exact propositions, three of which now rest on the one premise.

1.4 Status and plan

Every result carries one status: *proved here* (the proof is on the page), *proved in* a cited paper, *proved in work not yet published*, *exact computation* (a finite count done in full), *measured*, or *open*. Russell's claims are attributed to him and quoted with the page of the 1926 edition. His 1930 essay [2] has no page numbers in the text we used and is cited without them.

Section 2 sets out the objects. Section 3 proves the main theorem, Section 4 the structure of the breath, and Section 5 its relation to three dimensions. Section 6 states the wall, places the breath against it, and shows that one premise on RS's own order of accounts yields it. Section 7 covers clocks and registers, and Section 8 the open propositions. Section 9 tells how we got here. Appendices hold the census statistics and the full table of Russell's principles.

2 Setting

2.1 Postings, windows and orientation

A *posting* is one unit entered in the ledger with a sign: + opens a unit (a debit) and − closes one (a credit). A *window* is a run of eight consecutive postings, written $w = (w_0, \dots, w_7)$ with each $w_i \in \{+, -\}$. Order the signs by $- < +$.

Definition 1 (Balanced window)**.** A window is *balanced* when exactly four of its postings are +.

Definition 2 (Steady window)**.** A window is *steady* when it is monotone: either $w_0 \geq w_1 \geq \dots \geq w_7$ or $w_0 \leq w_1 \leq \dots \leq w_7$. Equivalently, the + postings form one run and the − postings form one run.

Definition 3 (Tones)**.** The *tones* of a window are its running totals $\tau_t = \#\{i < t : w_i = +\} - \#\{i < t : w_i = -\}$ for $t = 0, 1, \dots, 8$.

The *orientation flip* reverses every sign: $\bar{w}_i = +$ exactly when $w_i = -$. It exchanges debits and credits, and on ratios it exchanges r and $1/r$. Balance and steadiness are both preserved by it.

2.2 Accounts, loans and equal units

The ledger has accounts $0, 1, \dots, N-1$. Each account has a debit column and a credit column, so a *record* is a list of counts $(d_k, c_k)_{k<N}$ of natural numbers, and the *skew* of account k is $\varphi_k = d_k - c_k$. A posting adds one unit to one column of one account. A *loan* is a unit opened on an account by a debit and later closed on the same account by a credit. Loans have *equal units* when every loan stays open for the same number of postings. Closing them in the order they were opened is *first in, first out.*

The *state* of the ledger after t postings is the set of accounts holding an open unit, written as a pattern $s_t \in \{0, 1\}^N$ with $s_t(j) = 1$ when account j holds one. A *renaming* is a permutation of the accounts. It acts on records, states and postings by relabelling the account index.

2.3 The journal

Section 6 uses the ledger in its journal form. An *event* $e = (s, t, r)$ has a source site $s \in \mathbb{N}$, a target site $t \in \mathbb{N}$ and a ratio $r > 0$. Its *reciprocal* is $e^* = (t, s, 1/r)$, so $e^{**} = e$, and its *cost* is $J(r)$, with $J(r) = J(1/r)$. A *journal* is a finite sequence of events; posting e appends it. A journal is *balanced* when every event occurs in it as often as its reciprocal. A posting e after a journal l is *completable* when some continuation m makes the journal $l \cdot e \cdot m$ (the journal l, then e, then m) balanced. A *law of content* is a rule $A(l, e)$ saying which events may be posted after the journal l, and it *selects content* when some journal admits one event and rejects another. A *renaming of sites* is an injective map $\sigma : \mathbb{N} \to \mathbb{N}$, acting on events by $\sigma(s, t, r) = (\sigma s, \sigma t, r)$.

3 The main theorem

Theorem 4 (The breath is unique up to orientation)**.** *A window is balanced and steady exactly when it is*

$$++++----\qquad \textit{or} \qquad ----++++.$$

The second is the orientation flip of the first. Status: proved here.

Proof. A steady window with its + run first is $+^{k}-^{8-k}$ for some k with $0 \le k \le 8$, and one with its − run first is $-^{8-k}+^{k}$. Balance says $k = 4$ in both cases. Conversely, both windows have four postings of each sign, and each is monotone. Reversing every sign of the first gives the second. □

Corollary 5 (Tones)**.** *The tones of* $++++----$ *are* $0, 1, 2, 3, 4, 3, 2, 1, 0$*, and those of* $----++++$ *are their negatives. Status: proved here.*

Proof. The running total rises by one at each of the first four postings and falls by one at each of the last four. □

The two conditions do real work together, and neither does it alone. Of the $\binom{8}{4} = 70$ balanced windows, two are steady. Of the sixteen steady windows (nine with the + run first, nine with the − run first, less the two constant windows counted twice), two are balanced. These are exact computations from the definitions.

Russell's tones run the same course. His octave is the series $4^{\pm}\ 3^{+}\ 2^{+}\ 1^{+}\ 0\ 1^{-}\ 2^{-}\ 3^{-}\ 4^{\pm}$, in which "Tone 4 is a double tone which is neither positive nor negative" [1, p. 82]. In the corollary the tone climbs from rest to four and returns. Russell calls the two ends "the two passing points where the opposing forces are in equilibrium" and adds that "At these two passing points all effects of motion are reversed" [1, p. 136]. In the breath the direction reverses at exactly those two tones, 0 and 4.

4 The structure of the breath

4.1 Equal units force the half window

Proposition 6 (Half window)**.** *Let four loans open at postings* $0, 1, 2, 3$ *of a window, and let each close* d *postings after it opened, within postings* 4 *to* 7*. Then* $d = 4$*: every loan stays open exactly half the window. Status: proved here.*

Proof. The loan opened at posting 0 closes at posting $d \ge 4$, and the loan opened at posting 3 closes at posting $3 + d \le 7$. So $d = 4$. □

So in the plus-first window, the loan opened at posting i closes at posting $i + 4$: first in, first out. When the four loans sit on four different accounts, name the accounts $0, 1, 2, 3$ in the order they are opened.

Definition 7 (The breath)**.** The *breath* on four accounts debits accounts $0, 1, 2, 3$ in turn and then credits accounts $0, 1, 2, 3$ in turn, and repeats. After t postings, account j holds an open unit exactly when

$$s_t(j) = 1 \iff j < (t \bmod 8) \le j + 4.$$

Table 1 lists the eight states, and Figure 1 draws them.

Table 1. The breath on four accounts. Posting t is the step from state $t-1$ to state t. "Held" is the number of postings for which the changed account had kept its previous state.

State t	$s_t(0)$	$s_t(1)$	$s_t(2)$	$s_t(3)$	Tone	Posting t	Held
0	0	0	0	0	0	credit 3	4
1	1	0	0	0	1	debit 0	4
2	1	1	0	0	2	debit 1	4
3	1	1	1	0	3	debit 2	4
4	1	1	1	1	4	debit 3	4
5	0	1	1	1	3	credit 0	4
6	0	0	1	1	2	credit 1	4
7	0	0	0	1	1	credit 2	4

Theorem 8 (The structure of the breath)**.** *In the breath on four accounts:*

(a) exactly one account changes at each posting;

(b) the account changed by the posting from state t to state $t+1$ is $t \bmod 4$, and it is the account whose state has stood longest: its last change was four postings earlier, against three, two and one postings for the other three accounts;

(c) the eight states $s_0, \dots, s_7$ are distinct;

(d) the number of open units runs $0, 1, 2, 3, 4, 3, 2, 1$, the tones of Corollary 5;

(e) the breath has period eight: $s_{t+8} = s_t$.

Status: proved here.

Proof. (e) holds because s_t depends on t only through $t \bmod 8$. For the rest, read t modulo 8. Account j changes between states t and $t+1$ exactly when $t+1$ crosses one of the bounds in Definition 7, that is, when $t + 1 \equiv j + 1$ or $t + 1 \equiv j + 5 \pmod 8$, which is when $t \equiv j \pmod 4$. For each t exactly one $j \in \{0, 1, 2, 3\}$ satisfies this, which gives (a) and the first half of (b). The changes of account j fall at the postings into states $j+1$ and $j+5$, four postings apart, so the account changed by posting $t+1$ last changed four postings earlier. The other three accounts are $t-1$, $t-2$ and $t-3$ modulo 4; they changed one, two and three postings earlier. This is the second half of (b). For (c), the set of open accounts is empty at $t = 0$, is $\{0, \dots, t-1\}$ for $1 \le t \le 4$, and is $\{t-4, \dots, 3\}$ for $5 \le t \le 7$; these eight sets are different. Counting them gives (d). □



Figure 1. The breath over eight postings. A filled cell means the account holds an open unit. Each posting changes one account, the one whose state has stood longest; the tone climbs from 0 to 4 and returns, and state 8 equals state 0.

In switching theory this sequence of states is the four-stage twisted-ring counter (the Johnson counter). What the theorem adds is that it is forced: Theorem 4 and Proposition 6 leave no other window with equal units on four accounts. The rule of part (b) is the act in one sentence. At each posting the account that has held its state longest changes it. The breath reads its own record, oldest entry first.

4.2 Opening and closing

The return half has an exact effect on the journal. Write l^* for the sequence of reciprocals of l, in the same order.

Proposition 9 (Open, then close)**.** *For every finite sequence of events l, the journal $l \cdot l^*$ is balanced, and its closing half carries the inverse ratios: the ratio of the ith closing posting is the reciprocal of the ratio of the ith opening posting. Status: proved here.*

Proof. Reciprocation is an injective involution, so the number of times an event f occurs in l^* equals the number of times f^* occurs in l. Hence f occurs in $l \cdot l^*$ as often as f in l plus f^* in l, and f^* occurs in $l \cdot l^*$ as often as f^* in l plus f in l. The two counts agree. The ratio of e^* is $1/r(e)$. □

5 Three dimensions

5.1 The breath's period and the cube

The breath on $D+1$ accounts debits each account once and credits each once, so its period is $2(D+1)$. The cube of D binary coordinates has 2^D corners, and RS's completion clock shows all of them in one pass of 2^D ticks, one coordinate changing per tick.

Theorem 10 (Only three dimensions)**.** *For natural numbers D, $2(D+1) = 2^D$ exactly when $D = 3$. Status: proved here.*

Proof. For $D = 0, 1, 2, 3$ the two sides are $2, 1$; $4, 2$; $6, 4$; and $8, 8$. For $D \geq 4$, $2^D > 2(D+1)$: it holds at $D = 4$ ($16 > 10$), and if $2^D > 2(D+1)$ then $2^{D+1} > 4(D+1) \geq 2(D+2)$. □

So one breath lasts exactly one pass of the cube clock only in three dimensions. The two counts are already part of the ledger's structure: in three dimensions the record frame has $D+1 = 4$ accounts, and the eight of shell capacity, $2(D+1)$, meets the eight of the tick, 2^D, only at $D = 3$ (proved in work not yet published). The breath gives the first eight an order in time: four accounts, each posted twice. Russell's octave says the same in his words, four units of two opposed tones each.

5.2 Read through three bits, the breath walks the cube

The breath's four-bit state shows eight of the sixteen four-bit patterns. Merge the two accounts that are opposite in its order, 0 and 2, into one bit.

Definition 11 (Three-bit reading)**.** The three-bit reading of the breath is $c_t = \big(s_t(0) \oplus s_t(2),\ s_t(1),\ s_t(3)\big) \in \{0,1\}^3$, where $\oplus$ is addition modulo two.

Theorem 12 (The walk through the cube)**.** *The readings $c_0, \ldots, c_7$ are the eight corners of the cube $\{0,1\}^3$, each once; consecutive readings differ in exactly one bit; $c_{t+8} = c_t$; and the first bit changes at every second posting, so the bits change in the order $a\,b\,a\,c\,a\,b\,a\,c$ with a the first bit, b the second and c the third. Status: proved here.*

Proof. From Table 1, the readings for $t = 0, \ldots, 7$ are

$$000,\ 100,\ 110,\ 010,\ 011,\ 111,\ 101,\ 001,$$

eight different patterns, and $c_8 = c_0$ by Theorem 8(e). Consecutive readings differ in the first, second, first, third, first, second, first and third bit in turn, including the step from $c_7 = 001$ back to $c_0 = 000$. □

The merged bit is where the cost of the reading falls. Merging accounts 0 and 2 turns the breath into a breath on three accounts, with postings on accounts $0, 1, 0, 2$ (renaming account 3 as 2) followed by credits in the same order. Read that way, the merged account receives two debits before its first credit.

Proposition 13 (The merged account holds two units)**.** *In the three-account form of the breath, with postings debit 0, debit 1, debit 0, debit 2, credit 0, credit 1, credit 0, credit 2, account 0 holds two open units after the third and after the fourth posting; every unit still closes four postings after it opened; and the parity pattern of the three accounts runs through all eight corners of the cube, one bit per posting. Status: proved here.*

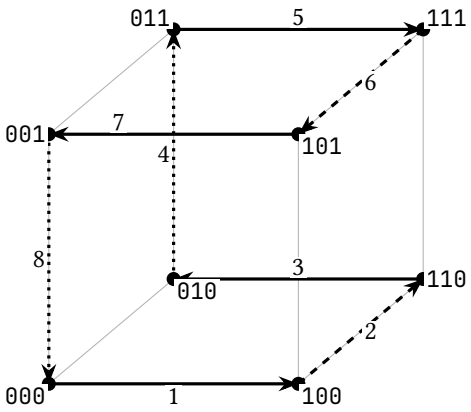


Figure 2. The breath read through three bits (first bit: accounts 0 and 2 merged; second: account 1; third: account 3). The walk starts at `000`; the number on each step is the posting. It visits every corner once. Solid steps change the first bit, dashed steps the second, dotted steps the third; the first bit changes at every second posting.

Proof. Account 0 is debited at postings 1 and 3 and first credited at posting 5, so it holds two units after postings 3 and 4. The loans open at postings $1, 2, 3, 4$ on accounts $0, 1, 0, 2$ and close at postings $5, 6, 7, 8$ on the same accounts. The parity of account k is the parity of its total number of postings; it changes whenever account k is posted. So the parity pattern changes in bit $0, 1, 0, 2, 0, 1, 0, 2$ in turn, which is the sequence $abacabac$ of Theorem 12, and it walks the same eight corners. □

Whether any account of the ledger ever holds two units decides whether the three-bit walk is the ledger's own record or a reading of a four-account ledger. Under the premise of Section 6.4 no account ever does, so on that premise the walk is a reading (Lemma 24, P2).

6 The act at the wall

6.1 The wall, stated exactly

The objects are those of Section 2.3. The first theorem says that the ledger's premises put no condition on the next posting, and that the only filter they offer, balance, is a filter on completions and not on single postings.

Theorem 14 (The premises never pick the next posting)**.**

(a) After every journal l, every event e is completable.

(b) Let l be balanced. The events $e_1 = (0, 1, 2)$ and $e_2 = (2, 3, 2)$ are different, neither is its own reciprocal, both are completable after l, both cost $J(2) = \frac{1}{4}$, and posting either lengthens the journal by one.

(c) The law that admits every event selects no content.

(d) If l is balanced and $e \neq e^*$, then $l \cdot e$ is not balanced. So the law "stay balanced after every posting" admits no loan.

Status: proved here; first proved in work not yet published.

Proof. (a) Take $m = (le)^*$; then $l \cdot e \cdot m$ is balanced by Proposition 9. (b) The two events have different sources. The reciprocal of $(s, t, 2)$ has ratio $\frac{1}{2} \neq 2$. Completability is (a), and $J(2) = \frac{2+1/2}{2} - 1 = \frac{1}{4}$. (c) No journal admits one event and rejects another. (d) In $l \cdot e$ the event e occurs once more than in l, while e^*, being different from e, occurs as often as in l, which by balance is as often as e occurs in l. □

The two witnesses of (b) are one event written twice: e_2 is e_1 with every site moved up by two. The next theorem shows that this is the general case, up to one bit.

Theorem 15 (What is left is a renaming and an orientation)**.**

(a) For $x, y > 0$, $J(x) = J(y)$ exactly when $y = x$ or $y = 1/x$.

(b) Renamings of sites preserve cost, commute with reciprocals, and carry balanced journals to balanced journals and completable postings to completable postings.

(c) If two events each join two different sites and have the same ratio, some renaming of sites carries one to the other.

So two equally cheap next postings differ by a renaming of sites, by the orientation (r against $1/r$), or by both. Status: proved here; first proved in work not yet published.

Proof. (a) $J(x) - J(y) = \frac{1}{2}\big(x - y + \frac{1}{x} - \frac{1}{y}\big) = \frac{(x-y)(xy-1)}{2xy}$, which vanishes exactly when $x = y$ or $xy = 1$. (b) Cost reads only the ratio, which a renaming keeps, and $\sigma(e^*) = (\sigma t, \sigma s, 1/r) = (\sigma e)^*$. Since σ is injective on events, an event σf occurs in the renamed journal as often as f occurs in the original; an event outside the image of σ has its reciprocal outside it too, and both occur zero times. So balance is kept, and renaming a balancing continuation keeps completability. (c) Given ends $s_1 \neq t_1$ and $s_2 \neq t_2$, a permutation of $\mathbb{N}$ sends s_1 to s_2 and t_1 to t_2. □

The theorem classifies equally cheap candidates. The premises leave more open than that. Nothing fixes the size of the ratio, nothing says whether the next event meets sites already posted or fresh ones, and nothing says whether to open another loan or close an open one; by Theorem 14(a) every choice is completable. A law that treats all sites alike cannot pick sites on the empty journal either: if it admits e there, it admits every renaming of e. The first posting names the sites and the direction, and only after it does the journal itself break the symmetry. So a law that picks the next posting must read the journal's history.

Proposition 16 (A static cost picks rest)**.** *The law "post a completable event of least cost" admits, after every journal, exactly the events of ratio* 1*. Status: proved here.*

Proof. By Theorem 14(a) every event is completable after every journal. By (1), $J(r) \geq 0$ with equality exactly at $r = 1$. □

The least-cost law does not read the journal at all, and what it admits costs nothing. It does permit loans in the sense of Section 2.3: the event $(0, 1, 1)$ joins two different sites, so it is not its own reciprocal, and it is completable. But every loan it permits has ratio 1 and compares a thing with its equal. A static cost and its strict minimum select rest, not a motion. On RS's clock carrier its own least-cost dynamics reaches its minimum in one step and then never moves (proved in work not yet published).

The last part of the wall concerns fields. The action of a posting from site p to site q under a real field ε on the sites is the cost of the ratio $e^{\varepsilon(p)-\varepsilon(q)}$, and by (1)

$$J\big(e^{z}\big) = \cosh z - 1.$$

Proposition 17 (No common action field)**.** *Fix $\hbar > 0$. On the period-three torus $(\mathbb{Z}/3)^3$, call a field ε a common action field if every primitive posting, from a site to its neighbour one step along an axis, carries one quantum:* $\cosh\big(\varepsilon(p) - \varepsilon(q)\big) - 1 = \hbar$*. Then each single posting has a field that carries one quantum on it, and no common action field exists. Status: proved here; the native action and its quantum are proved in work not yet published.*

Proof. The condition says $|\varepsilon(p) - \varepsilon(q)| = a$ with $a = \operatorname{arcosh}(1 + \hbar) > 0$. For one posting, put $\varepsilon = a$ at p and 0 elsewhere. For all of them at once, take three postings along one axis, $p_0 \to p_1 \to p_2 \to p_0$; on the period-three torus they return to the start. The three differences $\varepsilon(p_0) - \varepsilon(p_1)$, $\varepsilon(p_1) - \varepsilon(p_2)$ and $\varepsilon(p_2) - \varepsilon(p_0)$ sum to zero, yet each is $\pm a$, and a sum of three terms $\pm a$ is an odd multiple of a. □

Each act has its own field. The next posting cannot be read off one global potential, and whatever picks it is indexed by the event.

6.2 The breath as a law of content

Definition 18 (The breath law)**.** Fix a unit $u > 1$. The breath law B_u admits e after the journal l when e has ratio u if the length of l is $0, 1, 2$ or 3 modulo 8, and ratio $1/u$ if it is $4, 5, 6$ or 7 modulo 8. Sites are left free; by Theorem 15 they carry no content.

Reading a ratio above one as +, the breath law is Theorem 4's plus-first window posted in every window of eight.

Theorem 19 (The breath selects, and permits loans)**.**

(a) B_u selects content: after the empty journal it admits $(0, 1, u)$ and rejects $(0, 1, 1/u)$.

(b) After every balanced journal whose length is a multiple of eight, B_u admits an event that is not its own reciprocal and is completable.

(c) For every journal l_0 and every opening half l, the journal $l_0 \cdot l \cdot l^*$ has, for every event f, the same count of f minus the count of f^* as l_0. In particular, a window whose closing half is the reciprocals of its opening half, in the same order, keeps a balanced journal balanced.

Status: proved here.

Proof. (a) The empty journal has length 0, so only ratio u is admitted, and $1/u \neq u$ because $u > 1$. (b) The event $(0, 1, u)$ is admitted; it is not its own reciprocal, since that would need $u = 1/u$; and it is completable by Theorem 14(a). (c) Counts add over concatenation, and in $l \cdot l^*$ every event f occurs as often as f^* (Proposition 9), so appending $l \cdot l^*$ adds the same number to the count of f and to the count of f^*. □

Table 2. Four laws of content against the wall (Theorems 14 and 19, Proposition 16).

Law	Selects content	Permits loans	Reads the journal
Admit every event	no	yes	no
Stay balanced at every posting	yes	no	yes
Least cost	yes: ratio 1 only	only of ratio 1 (rest)	no
The breath B_u	yes	yes	yes: its length

6.3 What the breath leaves free

On the account record the breath fixes almost everything the wall leaves open: the sign of every posting, that loans open before they close, that every unit is the same size, that the four loans sit on four different accounts, and when each closes. What it does not fix is exactly two things.

Proposition 20 (The breath's freedom is a renaming and an orientation)**.** *On four accounts, call a window of eight unit postings a breath when its signs are balanced and steady, its four opening postings fall on four different accounts, and each unit closes four postings after it opened, on the account where it was opened. There are exactly* 48 *breaths. The renamings of the accounts together with the orientation flip form a group of* 48 *elements that acts on them, and any breath is carried to any other by exactly one element of the group. Status: proved here.*

Proof. By Theorem 4 the signs are $++++----$ or $----++++$. The opening postings fall on accounts a_0, a_1, a_2, a_3, an ordering of the four accounts, and the closing postings fall on a_0, a_1, a_2, a_3 in the same order. So a breath is fixed by its orientation and its ordering: $2 \times 24 = 48$. A renaming π sends the ordering (a_i) to (πa_i) and keeps the orientation; the flip changes the orientation and keeps the ordering. Renamings act on orderings with exactly one renaming carrying any ordering to any other, and the flip acts in the same way on the two orientations. □

The proposition tests a reading we had not checked: that what the breath leaves free is the freedom the wall already leaves. At the level of postings it holds exactly. Encode a plus on account a as the event (a, a, u) and a minus as its reciprocal $(a, a, 1/u)$; this encoding is a reading. Then a renaming of the accounts acts on the encoded postings as the wall's renaming of sites (extended by the identity to the other sites), and the orientation flip acts as the wall's reciprocal. The size of the unit u is the one further thing the breath does not fix, and it does not need to: a breath with equal units has one size throughout, and setting that size is RS's scale, not the order of postings.

At the level of the three-bit clock the reading fails, because the clock sees less than the postings.

Proposition 21 (The three-bit clock sees less)**.**

(a) Read through the three-bit reading of Definition 11, a breath and its orientation flip show the same patterns.

(b) Rename the breath so that account $\pi(j)$ is posted at phase j, and read it through the three-bit reading, which merges accounts 0 *and* 2*. The reading is a complete clock exactly when accounts* 0 *and* 2 *are posted at phases of the same parity. This holds for* 8 *of the* 24 *renamings; exchanging accounts* 1 *and* 2 *gives a breath whose reading is not complete.*

Status: proved here.

Proof. (a) The reading reads parities of postings, and exchanging debits and credits changes no parity. (b) Write $J_j(t) = 1$ when $j < (t \bmod 8) \le j + 4$, and 0 otherwise, for the state of the account posted at phase j. If accounts $0, 2$ sit at phases p, q and accounts $1, 3$ at phases r, s, the reading at tick t is $\big(J_p(t) \oplus J_q(t),\ J_r(t),\ J_s(t)\big)$. One tick of time sends J_j to J_{j-1} for $j \ge 1$ and J_0 to the complement of J_3: it moves the phases round cyclically and complements at most one coordinate, and neither changes whether the eight readings are different. So completeness depends only on whether p and q differ by two or by one, cyclically. If by two, shift to $\{p, q\} = \{0, 2\}$; the reading is the breath's own, up to an exchange of its last two bits, and is complete by Theorem 12. If by one, shift to $\{0, 1\}$; at $t = 0$ and $t = 2$ the merged bit $J_0 \oplus J_1$ is 0 and $J_2 = J_3 = 0$, so two readings coincide. The renamings with $\pi^{-1}\{0, 2\}$ equal to $\{0, 2\}$ or $\{1, 3\}$ number $2 \times 2 \times 2 = 8$. Exchanging 1 and 2 puts accounts 0 and 2 at phases 0 and 1. □

So the freedom the breath leaves is the wall's freedom on the postings, where all 24 renamings and the flip act. On the three-bit clock the flip is invisible, only a third of the renamings keep the clock complete, and the axis the clock singles out is the merged pair, which the reading chooses. This agrees with Section 7: a pass over a reading singles out the axis; the wall does not.

6.4 A premise that yields the breath

The two window conditions of Theorem 4 are conditions on signs. This subsection shows that one premise about each single posting forces both of them, up to the orientation, on the order of accounts RS already has. The premise is that no account ever holds more than one unit.

One posting is made per tick. The posting at tick t goes to account $\mathrm{acc}(t)$ with sign $\sigma_t = +1$ (a debit, a plus) or $\sigma_t = -1$ (a credit, a minus). The *held amount*, or balance, of account a after t ticks is $\varphi_a(t) = \sum\{\sigma_i : i < t,\ \mathrm{acc}(i) = a\}$, debits minus credits, and we write $\varphi(t) = \varphi_{\mathrm{acc}(t)}(t)$ for the holding of the account about to be posted. Posting four accounts *in turn*, $\mathrm{acc}(t) = t \bmod 4$, is RS's worked example of a legal clock on four accounts: one posting per tick, the accounts in turn.

Definition 22 (The one-unit bound; least held; opens plus)**.** The postings satisfy the *one-unit bound* when no account ever holds more than one unit: $|\varphi_a(t)| \le 1$ for every account a and tick t. They are *least held* when every posting takes the side that leaves the posted account's

held amount no larger in size than the other side would: $|\varphi(t) + \sigma_t| \le |\varphi(t) - \sigma_t|$ for every t. They *open plus* when a posting on an account that holds nothing is a plus: $\varphi(t) = 0$ implies $\sigma_t = +1$.

Least held is RS's own rule for the side of a posting. Read the cost on the ledger's state, $\sum_a J(u^{\varphi_a})$, the cost J of each account's balance after the posting; RS takes this reading at model status, that is, as a chosen reading of its cost and not a derived one. Then the side of least cost is the side that moves the posted account toward balance. It is unique exactly when that account is unbalanced, it respects the exchange of debits and credits, and on every history whose postings all take a least-cost side, from the empty ledger, every balance stays within one unit of zero. At an empty account the rule leaves the side open, and so must every rule that respects the exchange of debits and credits, because the exchange leaves an empty account as it is. RS's least-cost rules on the step itself, which read only the increments a posting adds, admit both sides at every ledger and pick nothing; that is why the side rule reads the balance. All of this is proved in work not yet published. On the account that the order names, least held is exactly this side rule (Proposition 23(a)), and the choice it leaves open at an empty account is exactly opening plus: the orientation, the residue the wall leaves (Theorem 15).

Proposition 23 (Three forms of least held)**.** *Fix a unit ratio $u > 0$ with $u \neq 1$. Each of the following is equivalent to least held:*

(a) each posting leaves the posted account's held cost $J(u^{\varphi})$, equivalently the state cost $\sum_a J(u^{\varphi_a})$, no larger than the other side would;

(b) for a function f strictly increasing on sizes, each posting leaves $f(|\varphi|)$ no larger than the other side would;

(c) each open unit costs $J(u)$, and each posting leaves the posted account's cost $|\varphi|\, J(u)$ no larger than the other side would.

Status: proved here.

Proof. $J(u^x) = \cosh(x \log u) - 1$, which is strictly increasing in $|x|$ because $\log u \neq 0$; that gives (a), and the state cost differs from the posted account's cost by the other accounts' costs, which the posting does not change. (b) holds because f is strictly increasing, and (c) is (b) with $f(k) = k\, J(u)$, since $J(u) > 0$. □

The drive's least-cost choice has the same form, read on registrations instead of balances; that likeness is a reading and carries no weight here.

Lemma 24 (One unit)**.** *Under least held, on any order of postings and any number of accounts, a posting on an account that holds nothing may take either side, and a posting on an account that holds one unit returns it. So every account holds at most one unit at every tick: $|\varphi_a(t)| \le 1$. Status: proved here.*

Proof. If $\varphi(t) = 0$, both sides leave size 1: a tie. If $\varphi(t) = \pm 1$, the side $\sigma_t = \varphi(t)$ leaves size 2 and the side $\sigma_t = -\varphi(t)$ leaves 0, so least held takes the return. From the empty ledger, induction on t. □

Proposition 25 (Least held is the one-unit bound)**.** *With one posting per tick, on any order and any number of accounts, from the empty ledger, the postings are least held exactly when they satisfy the one-unit bound. Status: proved here.*

Proof. Least held gives the bound by Lemma 24. Conversely, assume the bound. At tick t the posted account holds $\varphi(t) \in \{-1, 0, 1\}$. If $\varphi(t) = 0$ both sides leave size 1, so the posting is least held whatever its side. If $\varphi(t) = \pm 1$, the bound after the posting forbids $\sigma_t = \varphi(t)$, which would leave two units, so $\sigma_t = -\varphi(t)$, which is the least-held side. □

So the cost form adds an interpretation, not strength: least held is the one-unit bound stated as a cost rule. The premise of the next theorem is plainly that no account ever holds more than one unit. Read on the ledger, it is Russell's octave of four units, one to each account (a reading); and it is candidate 3 of Table 5, "no account holds more than one unit".

Theorem 26 (One premise yields the breath)**.** *Post four accounts in turn, one posting per tick, from the empty ledger. Every aligned window of eight postings is* ++++−−−− *exactly when no account ever holds more than one unit and the postings open plus. Equivalently (Proposition 25), exactly when the postings are least held and open plus. Status: proved here.*

Proof. Account a is posted at ticks $a, a+4, a+8, \ldots$ and at no others. Suppose the bound and opening plus. Account a holds nothing at its first posting. A posting from nothing leaves one unit, and the next posting of the same account, four ticks later, must return it, since the other side would leave two. So the postings of account a alternate: an opening at the ticks $a + 8m$ and a return at the ticks $a + 8m + 4$. Opening plus makes every opening a plus, so every return is a minus. For $a = 0, 1, 2, 3$ this puts a plus at ticks $8m, \ldots, 8m + 3$ and a minus at $8m + 4, \ldots, 8m + 7$: every aligned window is the breath. Conversely, if every aligned window is the breath, account a takes a plus at $a + 8m$ while holding nothing and a minus at $a + 8m + 4$ while holding one unit, so it never holds more than one unit, and every opening is a plus. □

Proposition 27 (The premise picks the side, not the account)**.** *After the breath's first posting, a debit on account* 0*, compare all eight single postings at tick* 1*, four accounts and two sides, with the cost read on each account's balance,* $\sum_a J(u^{\varphi_a})$*, or on the whole ledger's total,* $J(u^T)$ *with* $T = \sum_a \varphi_a$*. On either reading the credit on account* 0 *leaves cost* 0*, while the breath's debit on account* 1 *leaves two accounts open, at cost* $2J(u)$ *or* $J(u^2)$*, both positive. So the breath is least cost for the side on the account the order names, not among all eight postings. Status: proved here.*

Proof. The credit on account 0 returns every balance, and the total, to zero, and $J(1) = 0$. The debit on account 1 leaves $\varphi_0 = \varphi_1 = 1$ and $T = 2$, and $J(u) > 0$, $J(u^2) > 0$ because $u \neq 1$. □

Least cost among all eight postings gives other laws. Read on the whole total it gives the pair law and its mirror: over two ticks the moves are the sixteen blocks "a debit on one account, a credit on one account", and with ties weighed equally the four blocks that leave the place where it is carry weight $\frac{1}{4}$ together and each of the twelve transfers $\frac{1}{16}$. Read on each account's own balance, with the account chosen freely, it opens a unit and returns it at the next tick, so the place never moves (both proved in work not yet published). So the premise needs both of its parts. The order chooses the account, and the side rule chooses only the side. The order is part of what one tick is (Open proposition P5).

Proposition 28 (What carries the weight)**.**

(a) If the cost reads the debits plus the credits of the posted account, $n_a = d_a + c_a$, the two sides of every posting tie, and the rule decides nothing.

(b) If a tick may post nothing, then from the empty ledger posting nothing costs 0 *and every posting costs at least $J(u) > 0$, so the least-cost law never leaves the empty record.*

Status: proved here.

Proof. (a) A debit and a credit on account a each raise n_a by one, so they leave the same cost. (b) From the empty ledger any posting leaves one account holding one unit. □

So the result rests on two features: the cost reads a signed holding, debits minus credits, and every tick posts. RS's least-cost rules on the step are of the tying kind in (a), which is why they pick no side.

Corollary 29 (What the bound gives by itself)**.** *Under the one-unit bound (equivalently, least held), with four accounts in turn:*

(a) every account is square at every multiple of eight ticks, and every aligned window is balanced;

(b) the posting four ticks after an opening, on the same account, returns it: $\sigma_{8m+i+4} = -\sigma_{8m+i}$ for $i < 4$, first in, first out;

(c) the sign of each opening is free: the window $++-+--+-$, repeated, satisfies the bound and is balanced, and is not the breath.

Status: proved here.

Proof. The proof of Theorem 26 used opening plus only to fix the sign of each opening. Without it, account a still opens at $a + 8m$ and returns at $a + 8m + 4$, which gives (a) and (b), and the openings' signs are unconstrained, which gives (c). □

The return half in (b) is chosen by the posted account's balance alone. It does not read the record of what has been registered, and it does not need any pass to complete. A drive that registers at every tick and never completes runs alongside the breath without conflict (proved in work not yet published).

Corollary 30 (Exactly four accounts in turn)**.** *Under the one-unit bound (equivalently, least held):*

(a) from the empty ledger, if the first window posts one account twice among its first four ticks, the postings do not form the breath; so no order on three or fewer accounts, and not the order 0 1 0 1 2 3 2 3 *on four, carries the breath;*

(b) with n accounts posted in turn, some postings satisfy the bound, open plus and form the breath in every aligned window exactly when $n = 4$.

Status: proved here.

Proof. (a) From the empty ledger the breath's first four postings are pluses on accounts that start at zero; an account posted twice among them would hold two units, against the bound. Four postings on three or fewer accounts put two on one account. (b) For $n = 4$ this is Theorem 26. For $n \le 3$ it is (a). For $n \ge 5$ the posting at tick 4 goes to account 4, which has never been posted and holds nothing, so opening plus makes it a plus, while the breath's fifth posting is a minus. □

The breath lasts twice as many ticks as it has accounts, so four accounts in turn is a window of $2(D + 1)$ ticks, which is the cube's 2^D only at $D = 3$ (Theorem 10).

What the rule reads matters as much as the rule.

Proposition 31 (Read on the whole ledger, the same rule gives the pair law)**.** *Suppose instead that each posting takes the side that leaves the whole ledger's signed total $T(t) = \sum_{i<t} \sigma_i$ no larger in size, $|T(t) + \sigma_t| \le |T(t) - \sigma_t|$, with a plus whenever $T(t) = 0$. Then the postings alternate $+-+-\cdots$, $T(t) = t \bmod 2$, and no window is the breath. Status: proved here.*

Proof. The argument of Lemma 24, with the total in place of one account's balance, keeps T within one unit; from $T = 0$ the posting is a plus and from $T = 1$ it is the return. The breath's total reaches 4. □

Per account the rule gives the breath; on the whole it gives the two-tick pair law.

Proposition 32 (Two classes that cannot select the breath)**.**

(a) A property of windows that is unchanged by every permutation of a window's eight ticks, and holds of the breath, holds of the alternating window $+-+-+-+-$, which is balanced and not steady. Balance, counts, and every cost that reads only increments are such properties.

(b) A sign law that reads nothing, admitting the same signs at every tick, admits every window if it admits the breath.

Status: proved here.

Proof. (a) The alternating window is the breath with its ticks permuted. (b) Admitting the breath means admitting both signs. □

So a premise that selects the breath must read the record and its place in the pass. The bound on accounts in turn reads both: the order names the account from the tick, and the side reads that account's balance.

This is the wall's answer at its exact status. The content of every posting is fixed, up to the orientation, by one premise on the library's own order: no account ever holds more than one unit. The order names the account; the bound, which in its cost form is RS's own side rule read on balances, names the side; and opening plus names the orientation, which RS leaves open. What stays open is whether the ledger's accounts obey the bound, which RS has at model status (Open proposition P0), and the order, which is part of what one tick is (Open proposition P5).

6.5 What the breath does not settle

Three things about the breath stay open.

- **Whether the ledger's accounts hold at most one unit.** Equivalently, whether RS's side rule, with its cost read on balances, is the ledger's own; RS has that reading at model status. On four accounts in turn this is the same question as whether the ledger follows the breath, up to the orientation (Theorem 26; Open propositions P0 and P1). In the one four-state law of motion on file, the breath is not singled out: other runs of eight that return to balance carry more weight, and a debit followed by a credit on one account is not neutral in its weight there (proved in work not yet published). Under the bound the breath is forced, so that law's weights read something other than the posted account's balance.
- **What one tick is.** RS's worked example of a legal clock on the account record posts one unit per tick to the accounts in turn. That is the breath's order of accounts. The return half, the credits after the debits in the same order, is not in that example; the one-unit bound supplies it (Corollary 29). Whether the ledger's tick is that order, which register is its clock, and what one move of its law is, are the subject of Section 7, where they remain conditional.
- **The drive and the return.** A recognizer whose scope includes itself carries a standing deficit $J(n/r)$ between what it has posted and what it has registered. The deficit's argument moves at every tick, its least-cost choice registers the most it can, and its cost falls at every act that registers (proved in work not yet published). That drive fixes content up to ties in order and orientation, and the breath settles both ties. On a finite scope of eight patterns the drive goes silent once all eight are read, so it does not supply the return half. The one-unit bound does, from the posted account's balance, whether or not anything completes (Corollary 29); the drive chooses what is registered, and the side rule chooses the side.

7 Clocks and registers

This section collects what is known about reading the four-account record as a clock. Every result is conditional on what a register and a tick are taken to be. None of it settles what one tick of the ledger is.

7.1 Registers and clocks

A record is $x = (d, c)$ with $d, c \in \mathbb{Z}^4$, records add, and a *posting kind* is a pair (column, account) whose record is the unit vector in that entry. Write $\mathrm{tick}(x) = \sum_k (d_k + c_k)$ for the number of postings, $b_k = d_k - c_k$ for the signed place of account k and $n_k = d_k + c_k$. A *register* with b bits is a map R from records to $\{0,1\}^b = \mathbb{F}_2^b$.

- *R adds up over the record* when $R(x + y) = R(x) + R(y)$ for all records x, y. It need not take every pattern.

- *R names no account* when for every renaming π there is a permutation σ_π of the b bit positions with $R(\pi x) = \sigma_\pi\big(R(x)\big)$ for every record x. Renamings must act on the register by moving its bits. This is stronger than asking only that renamings respect which records share a value.
- A *run* is a sequence of records $n(0), n(1), \ldots$; it is *single-posting* when $n(t+1)$ is $n(t)$ plus one posting kind. A register reads a run as a cyclic pass of T patterns $R(n(t))$, $t \bmod T$.
- The pass is a *complete clock* when consecutive patterns, the last and the first included, differ in exactly one bit, and every one of the 2^b patterns occurs. A complete clock need not show each pattern only once. Its *least period* is the least $p > 0$ for which the repeated pass has period p.

Proposition 33 (The breath's registers)**.**

(a) The four-bit parity register $x \mapsto (n_k \bmod 2)_{k<4}$ adds up and names no account. Read through it, the breath changes one bit per posting but shows only eight of the sixteen patterns, so it is not a complete clock on this register.

(b) The three-bit reading $c(x) = \big((n_0 + n_2) \bmod 2,\ n_1 \bmod 2,\ n_3 \bmod 2\big)$ of Definition 11 adds up, and read through it the breath is a complete clock. It names a pair of accounts: a permutation σ of its bits with $c(\pi x) = \sigma(c(x))$ exists exactly for the four renamings that map the pair $\{0, 2\}$ to itself, and the renamings carry c to six registers, one for each pair of accounts. Debits on accounts 0 and 2 change the same bit.

(c) The pass that posts accounts $0, 1, 0, 2, 0, 1, 0, 2$, read through the parities of accounts $0, 1, 2$, is a complete single-posting clock that never posts account 3; its register names account 3.

Status: proved here.

Proof. (a) A renaming π moves bit k to bit πk. The breath's states are those of Table 1. (b) Completeness is Theorem 12. If π maps $\{0, 2\}$ to itself, $c(\pi x)$ is $c(x)$, with its last two bits exchanged when π exchanges accounts 1 and 3. If not, the first bit of $c(\pi x)$ is the parity of $n_a + n_b$ for a pair $\{a, b\} \neq \{0, 2\}$, which is none of the three bits of $c(x)$ as a function of the record (compare them on unit postings), so no σ exists. The renamings that map $\{0, 2\}$ to itself are the identity, the exchange of 0 and 2, the exchange of 1 and 3, and both together; so the orbit has $24/4 = 6$ members. (c) This is Proposition 13 read on four accounts; exchanging accounts 2 and 3 turns the register into the parities of accounts $0, 1, 3$, which is no permutation of its bits. □

7.2 Single-posting clocks

Theorem 34 (Single-posting clocks)**.** *Let R have b bits, add up over the record and name no account, and let a complete clock read a single-posting run through R. Then b is 1, 2, 4, 5 or 8, and every period of the clock is at least 2^b. Each value occurs, with least period exactly 2^b: $b = 1$, the parity of the tick; $b = 2$, the parities of $\sum_k d_k$ and of $\sum_k c_k$; $b = 4$, the parities of n_k; $b = 5$, the parities of the d_k with the parity of $\sum_k c_k$; $b = 8$, the parities of all eight entries. Status: proved here.*

Proof. At each tick the run adds one posting kind κ, and the pattern changes by $R(\kappa)$; since a complete clock changes exactly one bit per tick, $R(\kappa)$ is a unit vector for every kind the run posts. If some bit were the image of no posted kind, that bit would never change, and half the patterns would never occur. So the bits are exactly the images of the posted kinds.

Since $R(\pi\kappa) = \sigma_\pi R(\kappa)$ and σ_π carries unit vectors to unit vectors, if one debit kind is posted then all four debit kinds have unit-vector images, and the relation "debits on k and on l have the same image" is carried to itself by every renaming. Renamings carry any pair of different accounts to any other, so either all four debit kinds share one bit or they have four different bits. The same holds for credits. The relation "a debit on k and a credit on l have the same image" is likewise a union of the two renaming orbits of ordered pairs, $\{k = l\}$ and $\{k \neq l\}$. If the debits have four bits and the credits four, a coincidence for all $k \neq l$ would make a credit bit equal to three different debit bits, so the coincidences are none or exactly those with $k = l$. If one column has four bits and the other one bit, a coincidence would make the single bit equal to all four. Counting the bits: one column posted gives 1 or 4; both columns give 1 or 2 (one bit each, shared or not), 4 or 8 (four each, matched account by account or not), or 5 (four and one).

Every pattern occurs in the pass, so no period is shorter than 2^b. Each listed register adds up and names no account (a renaming moves the account-indexed bits like the accounts and fixes the others). For each, post along the reflected Gray code on b bits, at each step a kind whose image is the bit that changes; the pass shows each pattern once and has least period 2^b. □

Corollary 35 (No naming-free three-bit register takes every pattern)**.** *No three-bit register that adds up over the record and names no account takes every pattern. Status: proved here.*

Proof. Suppose R did. Then R is onto, so σ_π is fixed by π, and $\pi \mapsto \sigma_\pi$ is a homomorphism from the renamings to the permutations of three bits. The renamings it sends to the identity form a normal subgroup N whose quotient fits in a group of order six: N is the whole group, the even renamings, or the four-element group V of the identity and the three double exchanges. If N contains the even renamings, which carry any account to any other, all four debit kinds have one image and all four credit kinds have one image; the image of R is then spanned by two vectors and has at most four patterns. If N is V, the renamings that fix an account k meet V only in the identity, so they are carried onto all six bit permutations, and each fixes $R(\kappa)$ for the kinds κ on account k. A pattern fixed by every permutation of three bits is 000 or 111, so the image of R has at most two patterns. Either way R is not onto. □

Proposition 36 (A two-bit clock of least period eight)**.** *Read through the two-bit register (parity of $\sum_k d_k$, parity of $\sum_k c_k$), the run that posts on account* 0 *a credit, a credit, a debit, a credit, a credit, a credit, a debit and a credit shows the patterns* $00, 01, 00, 10, 11, 10, 11, 01$ *and then* 00 *again. It is a complete clock of least period eight that shows each of the four patterns twice. Status: proved here.*

Proof. Each credit changes the second bit and each debit the first, which gives the listed patterns; every pattern occurs, twice. Two debits and six credits return both parities. A period dividing eight and smaller than eight divides four, but the patterns at ticks 0 and 4 are 00 and 11. □

The same register also has the four-tick clock debit, credit, debit, credit.

Corollary 37 (Eight different states)**.** *If a single-posting complete clock shows eight different patterns, its register has three bits and, by Corollary 35, does not name no account. By additivity alone, either two different posted kinds change the same bit, or some account is never posted. Status: proved here.*

Proof. Every pattern occurs, so $2^b = 8$. Each posted kind changes one of three bits; if no two posted kinds change the same bit, at most three kinds are posted, and so at most three accounts. □

The breath's three-bit reading is the first case: debits on accounts 0 and 2 change the same bit. The pass of Proposition 33(c) is the second.

7.3 Clocks that read events: the registers S, T and F

An *event* is a record with an even tick. A register *reads events* when its two properties are required only on events, $R(x + y) = R(x) + R(y)$ and $R(\pi x) = \sigma_\pi(R(x))$ for events x, y, and the run it reads consists of events. It *takes every pattern on events* when every $v \in \mathbb{F}_2^3$ is $R(x)$ for some event x. Index three bits by the three ways of splitting the accounts into two pairs, {01|23}, {02|13} and {03|12}, and write A for the side of a pairing that contains account 0 and B for the other side. For events, define

$$S_p = \tfrac{1}{2}\Big(\sum_{k\in A} b_k - \sum_{k\in B} b_k\Big), \qquad T_p = \tfrac{1}{2}\Big(\sum_{k\in A} n_k - \sum_{k\in B} n_k\Big), \qquad F_p = \sum_{k\in A} d_k + \sum_{k\in B} c_k,$$

each modulo two. The halves are integers, since both differences have the parity of the tick. S reads the parities of the signed place's coordinates on the three axes, T the same from debits plus credits, and F the debits on one side of each pairing with the credits on the other. Write $g = \frac{1}{2}\,\mathrm{tick}(x)$ and $h = \frac{1}{2}\big(\sum_k d_k - \sum_k c_k\big)$, half the signed total, each modulo two.

Proposition 38 (The three registers)**.**

(a) S, T and F add up and name no account on events, with σ_π the permutation that π induces on the three pairings, and each takes every pattern on events.

(b) On events, $F = T + h\,(1,1,1)$ and $S = F + g\,(1,1,1)$.

(c) For a tick of two postings: S changes exactly one bit exactly when the two postings are in one column on two different accounts; T, exactly when they are on two different accounts, in any columns; F, exactly when they are a debit on one account and a credit on another, and those ticks keep the signed total. A debit and a credit on one account are unseen by S and change all three bits of T and of F.

Status: proved here.

Proof. (b) $T_p + h = \frac{1}{2}\big(\sum_A n_k - \sum_B n_k + \sum_k d_k - \sum_k c_k\big) = \sum_A d_k - \sum_B c_k \equiv F_p$, and $S_p + g = \frac{1}{2}\big(\sum_A (b_k + n_k) + \sum_B (n_k - b_k)\big) = \sum_A d_k + \sum_B c_k = F_p$. (c) Two postings on accounts $k \neq l$ in

one column change $\sum_A b_k - \sum_B b_k$ by ± 2 when k and l are on the same side and by 0 otherwise; exactly one pairing puts them on the same side, so S changes one bit. Two postings in one column on one account change every S_p; a debit on k and a credit on $l \neq k$ change S_p exactly for the two pairings that separate k and l; a debit and a credit on one account leave every b_k alone. The same count with n in place of b gives T: postings on two different accounts change one bit, and two postings on one account change all three. For F, a debit on k changes F_p when $k \in A$ and a credit on l when $l \in B$; a debit on k with a credit on l changes F_p exactly when k and l are on the same side, which is one pairing if $k \neq l$ and all three if $k = l$, while two debits or two credits on different accounts change two bits. (a) Each bit is a sum, or half a sum, of entries, so the registers add up on events. A renaming π carries the two sides of p to the two sides of πp. Exchanging the sides changes the sign of S_p and T_p, which leaves their parities alone, and changes F_p by the tick, which is even on events; so $R(\pi x)_{\pi p} = R(x)_p$. By (c) each bit is changed alone by some two-posting tick, so sums of such ticks reach every pattern. □

Theorem 39 (Exactly three)**.** *A three-bit register that adds up on events, names no account on events and takes every pattern on events agrees on events with S, T or F after a permutation of its three bits. Status: proved here; two steps are exact computations over the* 256 *maps from events to* $\{0, 1\}$ *that add up.*

Proof. Since R is onto, σ_π is fixed by π, and $\pi \mapsto \sigma_\pi$ is a homomorphism; as in Corollary 35, the renamings it sends to the identity form a group N that contains the even renamings or is V. Let f_1, f_2, f_3 be the three bits of R, maps from events to $\{0, 1\}$ that add up; there are exactly 256 such maps, $x \mapsto a \cdot x + \varepsilon \frac{1}{2} \operatorname{tick}(x)$ modulo two.

If N contains the even renamings, each f_i is unchanged by every even renaming. By exact computation the maps unchanged by every even renaming are the four sums of $\sum_k d_k$ and g (modulo two), and three bits taken from a two-dimensional space are dependent, so R would not be onto.

If N is V, the homomorphism sends to the identity exactly the renamings that the action of the renamings on the three pairings sends to the identity, so the two differ by an automorphism of the permutations of three bits; every such automorphism is a relabelling of the bits, so after a relabelling the homomorphism is that action: the bits are indexed by the pairings and $f_{\pi p}(\pi x) = f_p(x)$. Then $f = f_{\{01|23\}}$ is unchanged by the eight renamings that fix the pairing $\{01|23\}$, and f determines the other two bits. By exact computation the maps unchanged by those eight renamings are the eight sums of $F_{\{01|23\}}$, $\sum_k d_k$ and g. So $f_p = F_p + u$ for every p, with u a sum of $\sum_k d_k$ and g, a map unchanged by every renaming. On events $F_{\{01|23\}} + F_{\{02|13\}} + F_{\{03|12\}} = \sum_k d_k$, because account 0 is on the side A of all three pairings and every other account of exactly one. The F_p are independent by Proposition 38(a), so the only possible dependence among the $F_p + u$ is $\sum_p (F_p + u) = \sum_k d_k + u = 0$, that is $u = \sum_k d_k$. The remaining choices $u = 0$, $u = g$ and $u = g + \sum_k d_k = h$ give F, S and T, by Proposition 38(b) and $h = g + \sum_k d_k$ on events. □

The hypothesis that the run consists of events matters. On the record with one column (debits only), there is a register that adds up and names no account on events but names account 0 on records with an odd tick, with an eight-state complete clock along single postings (proved in work not yet published). So the older statement "the clock reads even records, hence its tick is two postings" needs the run to consist of events.

7.4 A complete pass singles out an oriented axis

Theorem 40 (Four, two, two)**.** *Let a pass of eight steps on the cube* $\{0,1\}^3$ *change one bit per step, return to its start, and visit all eight corners. Then one bit changes four times, at every second step, and the other two change twice each; read cyclically, the bits change in the order* $a\,b\,a\,c\,a\,b\,a\,c$ *for some labelling* a, b, c *of the three bits. Status: proved here.*

Proof. A closed pass changes each bit an even number of times. A pass that visits every corner changes each bit at least once, hence at least twice. Eight changes split into three even numbers each at least two only as $4 + 2 + 2$. Call the bit that changes four times a. Between consecutive changes of a the pass stays on one face of the cube where a is constant; there are four such stretches, alternating between the faces $a = 0$ and $a = 1$, and the two stretches on each face together visit its four corners once. A stretch of one corner would be entered and left along the same a-edge, revisiting the corner across it; so every stretch has at least two corners, and therefore exactly two. Each stretch then makes exactly one change of a bit other than a, so a changes at every second step. On each face the two stretches use two disjoint edges of a square, which are parallel, so they change the same bit; and the two faces use different bits, since otherwise the third bit never changes. This gives $a\,b\,a\,c\,a\,b\,a\,c$. □

The bit that changes four times is an axis of the cube, and the order of the other two is a sense of turning about it: projected onto the plane of b and c, the pass goes once round the unit square. So every complete pass singles out an oriented axis. Which one belongs to the pass, not to the register. Read through the breath's three-bit reading, the breath changes the bits $a\,b\,a\,c\,a\,b\,a\,c$, while single postings on accounts $1, 0, 1, 3, 1, 0, 1, 3$ change them $b\,a\,b\,c\,b\,a\,b\,c$.

The direction a pass singles out in the space of places is not fixed by the register either, even when two passes show the same patterns. Write a tick (a, b) for a debit on account a with a credit on account b, read through the register F (bits in the order {01|23}, {02|13}, {03|12}), and place the record at $P(x) = \big(\sum_A b_k - \sum_B b_k\big)_p$, the signed place's three pairing coordinates, which are twice the cube coordinates of Proposition 38. Take

$$\begin{aligned} &\text{pass 1:}\quad (0,1),(0,2),(2,3),(3,0),(1,0),(1,3),(3,2),(2,1);\\ &\text{pass 2:}\quad (0,1),(0,2),(1,0),(0,3),(0,1),(2,0),(1,0),(3,0). \end{aligned}$$

Proposition 41 (Same register, same patterns, two directions)**.** *Both passes are complete clocks of* F *with least period eight, and both show the patterns* $000, 100, 110, 010, 011, 111, 101, 001$ *in that order. After eight ticks both return every account to balance and the place to the origin. The places run through*

$$\begin{aligned} &\textit{pass 1:}\quad (0,0,0),(0,2,2),(2,2,4),(2,4,2),(0,2,2),(0,0,0),(2,0,-2),(2,-2,0);\\ &\textit{pass 2:}\quad (0,0,0),(0,2,2),(2,2,4),(2,0,2),(4,2,2),(4,4,4),(2,4,2),(2,2,0), \end{aligned}$$

and twice the vector area of the closed path of places, $\sum_t P_t \times P_{t+1}$*, is* $(-8,0,0)$ *for pass 1, along a cube axis, and* $(-8,8,8)$ *for pass 2, along a body diagonal. Status: exact computation.*

One register, one sequence of patterns, two different oriented directions: the direction a pass singles out is a property of the pass.

7.5 What a move is, under five conditions

Proposition 42 (Moves)**.** *Let a law of motion be a relation* $\mathrm{step}(x, y)$ *on records, and let its moves be the set M of records m with* $\mathrm{step}(0, m)$*, assumed nonempty. Impose five conditions: (i) one law at every record:* $\mathrm{step}(x, x + m)$ *holds exactly when* $\mathrm{step}(0, m)$ *holds, so that* $\mathrm{step}(x, y)$ *holds exactly when* $\mathrm{step}(0, y - x)$ *holds and every step adds a move; (ii) M is closed under every renaming of the accounts; (iii) a move posts and does not un-post: every entry of every* $m \in M$ *is at least* 0*; (iv) every move keeps the signed total* $\sum_k (d_k - c_k)$*; (v) every move makes two postings. Then M is one of three sets: the four blocks "a debit on k and a credit on k"; the twelve blocks "a debit on i and a credit on j" with* $i \neq j$*; or all sixteen. Status: proved here.*

Proof. By (i) every step adds a move, and by (iii) and (v) each move is two unit postings. By (iv) the numbers of debits and credits it adds are equal, so it is a debit on some account i and a credit on some account j. Renamings act on the pairs (i, j) with two orbits: the four pairs with $i = j$, and the twelve with $i \neq j$, since a renaming can carry any ordered pair of different accounts to any other. By (ii), M is a union of orbits. □

Condition (v) is put in by hand: without it, larger moves such as three debits and three credits satisfy the other four. None of the five conditions chooses among the three sets. A debit and a credit on one account leave the signed place where it is, and on two different accounts they move it; so "the place moves at every move" gives the twelve and "the place never moves" gives the four. On the place, the average step over the sixteen is one quarter rest plus three quarters the average over the twelve, so the place alone cannot tell the sixteen from the twelve with a rest added. The twelve are the ticks of the forward register F, and the four are the ticks the signed place does not see. The breath's own postings, read in pairs, are neither: its first half posts two debits at a time.

7.6 The tick is the wall

None of this decides what one tick of the ledger is. A single posting, a debit-and-credit block, and two postings in one column are each the one-bit tick of some register, and Theorems 34 and 39 say which registers go with which. Which register is the ledger's clock is open (Open proposition P5). So the wall, stated from the side of the act, is the tick. The breath is a result about a window of postings, conditional on what a posting is.

8 Open propositions

Each proposition is stated with its hypotheses. None is assumed anywhere in this paper. Section 6.4 reduces three of them to one premise, which is stated first.

Open proposition P0 (One unit)**.** No account of the ledger ever holds more than one unit. Equivalently, given one posting per tick, the ledger's postings are least held: each takes the side of least cost on the posted account's balance, $J(u^{\varphi})$ for any unit ratio $u \neq 1$ (Propositions 23 and 25). *Status:* open.

Its standing in RS: the side rule with the cost read on balances is RS's own, at model status, and RS proves that it keeps every balance within one unit; the side at an empty account,

the orientation, RS leaves open. Open propositions P1, P2 and P4 rest on P0, together with the order of four accounts in turn: on that order, P1 is equivalent to P0 with opening plus (Theorem 26); P2 asks for the opposite of P0 on its record, so under P0 its answer is no; and under P0 the return half of P4 happens (Corollary 29). The order in turn is part of what one tick is, and stays open with P5.

Open proposition P1 (The ledger follows the breath)**.** Read the ledger's postings as a sequence of signs $\sigma_0, \sigma_1, \dots$, with + for a ratio above one in a fixed orientation, and windows aligned at multiples of eight. Then for every k, $(\sigma_{8k}, \dots, \sigma_{8k+7}) = ++++----$. *Status:* open.

On four accounts in turn it is equivalent to Open proposition P0 together with opening plus (Theorem 26). The breath is the only balanced, steady window (Theorem 4). In the one four-state law of motion on file, other runs of eight that return to balance weigh more than the breath; under the one-unit bound that law's weights must read something other than the posted account's balance.

Open proposition P2 (Two units)**.** On the three-account record, starting from the zero ledger, with single postings scheduled so that every window of eight ticks shows all eight parity patterns (Open proposition P3) and every account's skew returns to zero every eight ticks: in the ledger's own postings some account holds two units at some tick. *Status:* open.

Under those hypotheses the accounts are posted in the order $abacabac$ (Theorem 40), and two side laws compete on three accounts. One unit per account keeps every skew between -1 and 1, so the tones reach only 3. Steady windows give the breath's signs, so the account posted four times takes two debits before its first credit and holds two units after the third and fourth postings (Proposition 13), and the tones reach 4. Under P0 the answer is no, since P2 asks for the opposite of the bound, and the breath cannot run on three accounts (Corollary 30). The fork then dissolves: on four accounts in turn, one unit per account and the breath hold together. The two units on the merged account belong to the three-bit reading, not to the four-account ledger. As an exact computation: on the twelve directed complete passes from the zero pattern, the assignments of debits and credits that return every account to zero number 24 per pass (the account changed four times takes two debits in $\binom{4}{2} = 6$ ways, the other two accounts one each in 2 ways), so 288 in all, and exactly two per pass carry the signs of a breath, 24 in all. The one-unit bound is a premise that reads how much an account holds.

Open proposition P3 (No beginning)**.** Call a process $w : \mathbb{N} \to \{0,1\}^d$ *phase-free* when every window of 2^d consecutive ticks shows all 2^d patterns. The ledger's clock is phase-free. *Status:* open.

Proposition 43 (Phase-free processes)**.** *A phase-free process has period 2^d, and its pattern at each tick determines its pattern at the next. Status: proved here.*

Proof. The windows starting at n and at $n+1$ share the $2^d - 1$ ticks from $n+1$ to $n + 2^d - 1$, which show $2^d - 1$ different patterns. Each window shows all 2^d, so $w(n)$ and $w(n + 2^d)$ are both the one pattern the shared ticks miss: $w(n + 2^d) = w(n)$. Within each period every pattern occurs once, so $w(i) = w(j)$ forces $i \equiv j \pmod{2^d}$ and then $w(i+1) = w(j+1)$. □

Russell writes “No state of motion ever began or ever ended” [1, law 24, p. 66]. His sentence is consistent with phase-freedom but does not give it: a process can repeat without end and still have a marked tick. For a complete clock, phase-freedom is RS’s open clock law (a successor fixed by the state, returning at eight) in a form that speaks only of what is read.

Open proposition P4 (The return half when the scope includes itself)**.** Let a recognizer’s scope include its own reading, so that no pass of its record completes. Then its postings contain a return half: after each run of debits, the same accounts are credited in the same order. *Status:* open.

Under the one-unit bound, with four accounts in turn, the return half happens whether or not the scope completes, because the side is read from the posted account’s balance and not from the record of what has been registered (Corollary 29). So P4 reduces to P0 with the order in turn.

Open proposition P5 (The clock names no account)**.** The ledger’s clock reads a run of events through a three-bit register that adds up and names no account on events and takes every pattern on events. *Status:* open.

If it holds, Theorem 39 makes the register S, T or F up to a permutation of its bits, and Proposition 38 gives its one-bit ticks, each of two postings; which of the three is a further open question. If the clock instead posts single units, Corollaries 35 and 37 make its register name accounts: two posted kinds share a bit, or an account is never posted, as with the breath’s three-bit reading. Neither side is derived. This is the precise form of the question what one tick of the ledger is. The order of four accounts in turn, on which Theorem 26 rests, belongs to the same question.

9 History and discussion: how we got here

The breath was not found by searching over windows. It came out of a systematic reading of Walter Russell, and the reading had a rule: Russell’s text proposes, and a statement of his stays in a result only where a measurement or a proof keeps it there. This section tells that history, with its counts, including what failed.

9.1 Russell’s two books

The Universal One (1926) is two books in one volume [1]. Book I states Russell’s principles in nineteen short chapters, from creation and the one substance to omnipresence, omnipotence and omniscience. Book II, the dynamics of Mind, builds a mechanics on them: electricity and magnetism, the wave and the cube of motion, the octave and the elements, gravitation and radiation. The volume closes with a numbered list of new laws and principles, 199 in all, each with the page where the body states it. Its Prelude puts the whole program in one sentence: “There is but One universe. One Mind, One force, One substance” [1, Prelude].

The Russell Genero-Radiative Concept, or The Cyclic Theory of Continuous Motion (1930) is an essay of about 4,600 words [2]. It restates the unity of force from the 1926 book and adds an experiment: bar magnets hung on threads around a larger magnet, which take up stable

positions at distances set by their density and turn when the central magnet turns. Section 9.10 says what it keeps and what it changes.

9.2 The census

We read the 1926 book in full, in ten parts. In each part every claim was recorded with its place in the book, Russell's exact words, its kind, its subject, its numbers and its terms. That gave 4,495 recorded claims. Every recorded quotation was then compared with the book's text, allowing for the scan's character errors: 2,163 matched exactly, 1,986 closely, 172 in part, 136 could not be found (most are words printed on chart plates, which the scan reads poorly) and 38 were not checked. Merging the parts removed repeats and left 4,047 distinct claims in eleven registers by subject, with a glossary of 292 of Russell's terms, 50 numbered lists and 893 numbers (Appendix A).

From the claims we reconstructed Russell's system in his own terms: sixteen first principles, the order in which he builds on them, and the place of each of his 199 laws. Five principles are primary (he gives them as attributes of Mind), seven are derived from earlier ones, and four are stated as structures or numbers without derivation. Every claim group and every law falls under the sixteen, and no smaller set covers them. Appendix B lists the sixteen with his words.

9.3 Russell against measurement

Of the 4,047 claims, 467 have a reading that measurement can decide; the rest are definitions, statements about Mind, or quantities with no measured counterpart. Each was checked against measurement with its sources. The verdicts are in Table 3 and Figure 3.

Table 3. Russell's measurable claims against measurement, by subject. A: agrees; R: agrees on one reading; P: agrees in part; D: disagrees; U: cannot be decided as stated.

Subject	Claims	A	R	P	D	U
Light	32	7	5	8	11	1
Electricity and magnetism	48	5	2	3	38	0
Gravitation and radiation	45	8	2	7	28	0
The universe and the solar system	92	21	2	17	52	0
Energy and heat	27	4	3	5	12	3
The elements	95	24	14	20	36	1
Chemistry, colour and tone	84	36	8	17	22	1
Motion and the wave	27	6	1	2	12	6
Space, number and time	12	1	0	6	5	0
Mind, life and appearance	5	2	3	0	0	0
Total	467	114	40	85	216	12

The pattern is consistent. Russell is reliable where he reports the measurements or arrangements of his day: the periodic grouping of the elements and their principal valences, the

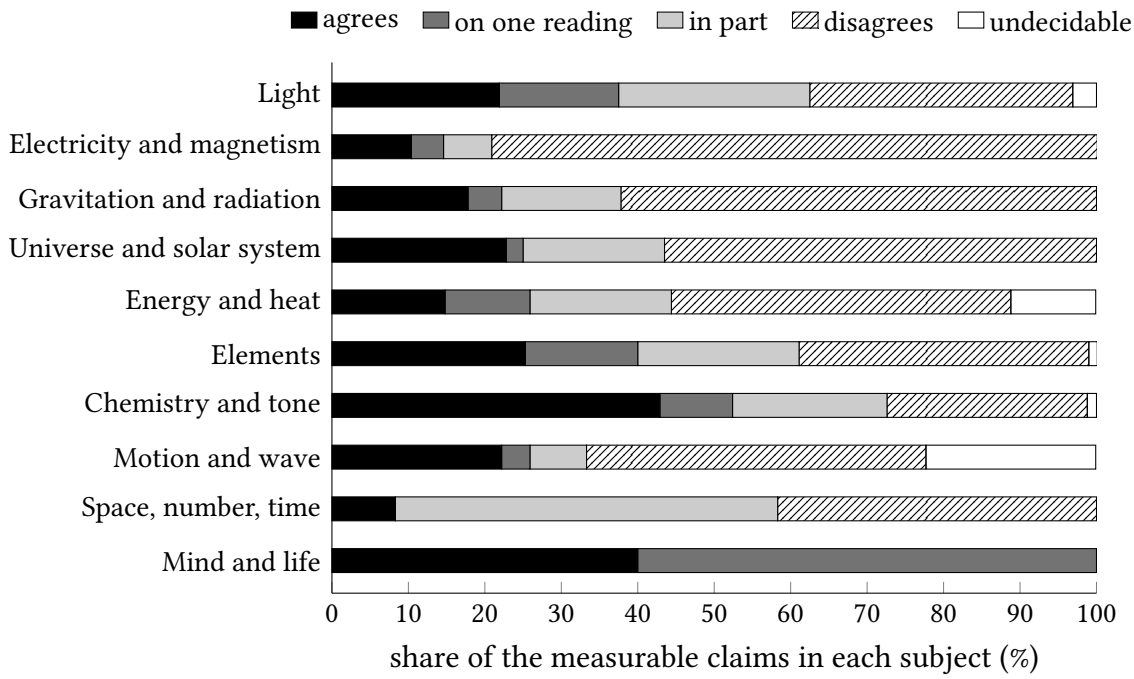


Figure 3. Russell's measurable claims by subject, as shares of each row of Table 3. Electricity and magnetism is his weakest subject; chemistry, where he reports the periodic grouping and valences of his day, is his strongest.

solar-system data he quotes, Kepler's second law, the heating of a contracting gas, and the speed of light, which he gives to within a few hundredths of a percent. He fails where his own laws go beyond those measurements. Electricity and magnetism is the weakest subject, with 38 of 48 claims against measurement: like charges repel and unlike charges attract, contrary to his laws, and electric and magnetic lines meet at right angles, not at the 180° he gives. His ten octaves of elements, his elements below helium, his speeds of light above c for higher octaves, his law of orbits by density and his instantaneous gravitation all fail.

9.4 His organizing structure is RS's

We then matched the sixteen principles to RS, principle by principle, with one verdict for each principle's organizing statement: *same* (RS states it), *same once RS's premises are attached*, *Russell says more* (he asserts what RS holds open), *RS is silent*, *in tension*, or *against measurement*. Table 4 gives the result after an independent check.

Two principles are RS's as they stand, and nine more once RS's premises are attached: eleven of sixteen. The two chains also run in the same order. Both begin with one thing dividing into a pair that sums to nothing, and both put the record next. Both have the same gap in the same place: Russell separates what is at rest from what moves it and gives the mover as a primary principle he does not derive, and RS separates the statics of the cost from the law of the step and holds the step open. Russell lacks one thing, a measure of imbalance. His balance is yes or no. The cost J says how far a ratio is from balance, and everything about ratio follows from it. That is why his magnitudes fail where his structure holds. His numbers are RS's theorems or premises: his three is the dimension D, the eight of his cube is 2^D, the eight of his octave is $2(D+1)$, his four is $D+1$, and his ratio of two is RS's join of two equal

Table 4. Russell's sixteen principles against RS. "Same, scoped" means the same once RS's premises are attached.

	Principle	Verdict	In RS
P1	The One substance	same, scoped	one carrier: a thing is its postings
P2	Thinking, the One force	RS silent	the mover: the wall
P3	Inertia: rest is balance	same	J has its one zero at ratio 1: zero is balance, not absence
P4	Desire to divide and to unite	same, scoped	the reciprocal pair; the balanced pair costs least
P5	The pulse of action and reaction	same, scoped	out and back: two half-steps per tick
P6	Balance, simultaneous, preponderant in sequence	same, scoped	double entry
P7	Registration: matter is the record	same, scoped	the record of postings
P8	Reproduction: light does not travel	Russell says more	light as the ledger's one-tick reach, open in RS
P9	Universal One-ness	same, scoped	one law at every scale
P10	Finitude	same, scoped	the kept record
P11	The universal constant of energy	same, scoped	balance; one quantum per posting
P12	The wave and the cube of motion	same	three dimensions; eight corners
P13	The octave	same, scoped	the valence octave
P14	The ten-octave cycle	against measurement	seven of his ten octaves are the measured periods
P15	The universal ratios	in tension	his 2 and RS's φ: equal and unequal join
P16	Potential position	against measurement	as a value of a ratio, same once scoped

parts, beside the golden ratio from a join of unequal parts. Take a ladder of levels $s_n = s_0 r^n$ with $s_0 > 0$ and $r > 0$. If a level is the sum of the two levels before it, $s_{n+2} = s_{n+1} + s_n$, then $r^2 = r + 1$ and $r = \varphi$. If a level is the sum of two copies of the level before it, $s_{n+1} = s_n + s_n$, then $r = 2$. Two equal parts taken two levels down, $s_{n+2} = s_n + s_n$, give $r^2 = 2$ instead; so "equal parts give 2" means parts from the level directly before (proved here: divide by $s_0 r^n$).

9.5 Light does not travel

"Light does not travel. It reproduces itself" [1, p. 30]. Read literally, the sentence fails measurement: light takes the time d/c to arrive, and it delivers measured energy and momentum. Russell's deeper statement is that a state of motion "is reproduced by union with its opposite effect at every inertial plane" [1, p. 182]: nothing is carried; the state is made again at each step. That is what light is: the ledger reproducing a state from one cell to the next at every tick, with no thing crossing. Whether light's speed is exactly the one-tick reach of the ledger is an open proposition in RS, and Russell's sentence, on this reading, asserts it. His light has no single speed, which measurement contradicts; RS's has one.

9.6 Where he says more, the subject is what moves

Five principles go beyond RS or against it: thinking as the mover (P2), reproduction (P8), the ten octaves (P14), the ratios (P15) and potential position (P16). Measurement removes the ten octaves, potential position as a place in space, and the literal reading of reproduction; the tension over the ratios dissolves inside RS, which has both. In what remains, and in every part of a principle where RS is silent, the subject is the same: what moves, in which direction and in what order. RS's wall is in the same place. We therefore wrote down Russell's own account of the mover in six statements:

- one force, whose control is two opposed desires: "Sex is the active desire of Mind for division into opposites, and its reactive desire for unity", the quality that "continues one impulse of thinking into the next impulse of thinking" [1, p. 23];
- every action is met by a reaction that restores the balance [1, p. 102];
- each thing seeks its true position and its more equal mate [1, pp. 28, 144];
- both opposites are present at once, each outweighs in turn, and the plus comes first: "All motion begins in the plus, contractive, endothermic impulse of thinking, and ends in the succeeding minus, expansive, exothermic impulse" [1, law 59, p. 117];
- no state of motion began or ended: "The end of a wave is but the beginning of another wave" [1, p. 135];
- at balance no force acts, and the motion is carried through by impetus: the dead centre "can only be passed by the impetus given to an object in motion prior to its arrival at that point" [1, p. 16].

From these we wrote fifteen candidate laws for the next posting, each in RS's terms, and put each against the wall. An independent check then reworked every candidate. Table 5 gives the verdicts after the check.

Nine stand: one as written and eight with corrections. Five are refuted, least cost among them, and one is faulty. The most useful survivor is the least committal: no beginning, the phase-free condition of Open proposition P3. The sharpest finding was a fork. On three accounts under phase-freedom, the side of each posting is decided either by one unit per account or by steady windows, and the two exclude each other. That is Open proposition P2. The fork later dissolved. On four accounts in turn, one unit per account is the breath's own premise (Section 6.4): the rival became the reason.

9.7 How the breath came out of that

Candidate 6, "all plus, then all minus", is Russell's law 26 read as runs: "each opposite is cumulatively preponderant in sequence" [1, law 26, p. 67]. In its first form it was refuted. With one unit on each of three accounts, debiting accounts $0, 1, 2$ and then crediting them gives six postings and six parity patterns, $000, 100, 110, 111, 011, 001$, and misses the corners 010 and 101; the pass is not complete.

Table 5. Fifteen candidate movers from Russell's text, after the independent check.

	Candidate	Verdict	Why
1	least cost as the selector	refuted	it admits only ratio 1: rest (Proposition 16)
2	the reaction closes the action at once	refuted	a post and its undoing visit at most $d + 1$ patterns
3	no account holds more than one unit	stands, corrected	an addition to Russell; it excludes the breath on three accounts, and on four accounts in turn it is the breath's premise (Theorem 26)
4	every opening is a plus	stands, corrected	its content is that every opening is on one side
5	impetus carries through the dead centre	stands, corrected	rival of 4: the breath with its orientation alternating by window
6	all plus, then all minus	stands, corrected	the breath; first judged refuted
7	the preponderant side picks the direction	refuted	corrected, it is the breath's rule
8	three to one, then one to three	stands, corrected	a theorem about pattern distance, not a selector
9	no beginning (phase-free)	stands, corrected	the next premise to try (P3)
10	rebound at the dead centre	stands, corrected	RS's own countermodel, not Russell's rebound
11	two opposed desires in turn	stands, corrected	candidate 9 written out as a rule
12	comparison of two held states	faulty	a form that excludes nothing
13	the more equal mate	stands	a theorem about settling, not a mover
14	the true position	refuted	its fixed form rests; its moving form only renames
15	impetus alone	refuted	one repeated step has parity period two

The repair came from Russell's octave. His octave has four units, not three: "There are four exactly equal unit constants of energy in each octave" [1, p. 82]. Four equal units each way is a window of eight postings, balanced, with each sign in one run. Stated that way, Theorem 4 says the window is unique up to orientation, and on four accounts Proposition 6 and Theorem 8 say it is one act. The independent check of the candidates then found the three-bit walk of Theorem 12: the same breath, with two opposite accounts read as one, completes on three accounts. That reversed the verdict on candidate 6, and it showed where the cost sits: the merged account holds two units at the turn.

Two corrections from a parallel line of our work belong to this history. RS's own worked example of a legal clock already posts to the accounts in turn, so the breath's order of accounts was in RS; its return half was not. And the clock results first reported in that work were corrected before this paper was written: Section 7 states the corrected versions.

The last step came from the open propositions themselves. Asking whether any account ever

holds two units, and why the return half happens, led to a premise that reads held amounts: least held. It turned out to be RS's own side rule, and, given one posting per tick, the same statement as the one-unit bound. It reduced three open propositions to one premise on RS's own order of accounts (Section 6.4), and it turned candidate 3, the one-unit rival that had excluded the breath on three accounts, into the breath's own premise on four.

9.8 The law of union and the boundary gate

RS's account of boundaries supplies one rule by hand, the gate: a contact between two sites posts only when the pair holds opposites; between equals it is silent and the state stays. Russell states the same rule as a law of union: "Unions of opposed actions and reactions are possible only within certain limitations. When union does not take place there can be no reproduction" [1, law 5, p. 28]. His law 6 adds that equal and opposite actions and reactions "when united, are satisfied in their unions and will remain united" [1, p. 28].

To make the match exact, let H be the finite-dimensional space of joint states of a contact's pair, let the native posting be a complete instrument $(K_j)_{j\in L}$ on H, so $\sum_j K_j^\dagger K_j = 1$, and let P be the orthogonal projection onto the states in which the pair holds opposites. Read Russell's law as four clauses on an instrument $(G_j)_{j\in L}$ with one silent outcome G_0: (U1) no posting acts between equals, $G_j(1-P) = 0$; (U2) where union occurs the posting is the native one, $G_jP = K_jP$; (U3) where union does not take place there is one silent outcome, an orthogonal projection, $G_0^\dagger = G_0 = G_0^2$; (U4) the instrument is complete, $\sum_j G_j^\dagger G_j + G_0^\dagger G_0 = 1$.

Theorem 44 (The gate from the law of union)**.** *Under (U1)–(U4), $G_j = K_jP$ for every j and $G_0 = 1 - P$: the instrument is the gate. Status: proved here.*

Proof. $G_j = G_jP + G_j(1-P) = K_jP$. Then $\sum_j G_j^\dagger G_j = P\big(\sum_j K_j^\dagger K_j\big)P = P$, so (U4) gives $G_0^\dagger G_0 = 1 - P$. For an orthogonal projection $G_0^\dagger G_0 = G_0^2 = G_0$, so $G_0 = 1 - P$. □

The projection clause matters: $G_0 = -(1-P)$ is self-adjoint and satisfies (U1), (U2) and (U4), but it is not a projection. And clause (U1) is a genuine premise, not a reading of the native instrument: the native posting does act on balanced pairs, through its postings with zero middle current (proved in work not yet published). So Russell's law forces the gate exactly, and the gate remains a premise. Of the 27 items RS's account of boundaries supplies by hand, Russell's text supplies none outright and four in part (the gate, the placement of contacts, the shared corner of eight cubes, and the opening of the books), cuts against one (separate subjects, which he denies: "Individuality is non-existent" [1, p. 40]), and fixes nothing on the other twenty-two.

9.9 What did not hold up

- **New territory.** We looked for structures of Russell's that RS has no object or question for, and found nine. Each was given a first theorem. Every first theorem is true, and every one is textbook mathematics: none is new content for RS. After correction, five leave medium-sized open questions, and three of those are not yet questions in RS at all: whether the light field's mean over a window is the static potential; stable against least-cost pairing of physical units; and how the return half of a persisting form is paired.

- **Discoveries at the level of principles.** The principle-by-principle match proposed seven discoveries. None survived as stated. Four were already in RS, two survived with corrections, and one was faulty.
- **Two against the golden ratio.** Russell's ratio 2 and RS's φ are not rivals. Both are in RS, as the equal and unequal branches of the join.
- **Ten octaves.** Ten octaves of elements fails measurement; seven of his ten are the measured periods of the table.
- **What his text anticipates.** Where his text anticipates RS, it anticipates RS's premises and constructions, such as the permanent record, and not its derived results: two clear anticipations, seven partial, three none.

9.10 The 1930 revision

We read the 1930 essay in full and recorded 130 claims, each compared with the 1926 census. Seventy-six restate the 1926 book, twenty-one sharpen a 1926 principle, three change one, and thirty are new; fourteen of the new ones report the magnet experiment.

- **Kept.** The unity of force, stated more sharply than in 1926: "I contend that electricity and gravitation are one, also that magnetism and radiation are one, and that both of these forces make up the two opposing forces evident in every effect of motion" [2]. The pulse is still one force in two directions, "One Force divided in its pulsations so that it appears to be two opposing forces" [2]. So are the charge doctrine, the seeking of a position proper to a potential, and work as divided pressure returning to equilibrium.
- **Sharpened.** Generation and radiation are mapped onto devices of his day: step-up and step-down transformers, the solenoid, and a room of air pumped into a tank. These add mechanisms and pictures, not numbers.
- **Changed.** Which force accelerates. In 1930 the inward, electric oscillation pumps force "with ever increasing acceleration" and magnetism is "that decelerative, repellant force" [2]; in 1926 generation was the deceleration of speed into power and magnetism the speed force. He keeps only the orbital sense, falling speeds up and rising slows.
- **New.** Apart from the experiment, one structure: "the combined triad effect of the two opposing oscillations and resistance is registered in waves of varying dimensions which are the basis of all motion" [2].
- **Left out.** The octave, its tones and locked potentials, the ten-octave cycle, inhalation and exhalation as the pulse of motion, light as the substance, and reproduction: the word "reproduce" does not occur, and energy "escapes" and "returns". The essay denies none of these; it does not use them.

So the two window conditions of this paper come from the 1926 book. The 1930 essay keeps the pulse but not its arithmetic of four units. Three of its sentences, all restatements of 1926, bear on the act:

- The return never completes: all motion is "an endless series of high and low pressure conditions which are ever interchanging in their constant and useless attempts to equalize. This can never be accomplished so long as the One Force is a living, pulsing force" [2]. This is the case of Open proposition P4, a scope in which nothing completes.
- The balanced point is passed by impetus: "No mass in the universe can keep such a position when it finds it, for the impetus of its motion in getting there always carries it past that point" [2].
- A discharge on one body is a charge on the next: "Mercury's negative discharge becomes positive charge for Venus" [2]. Read on the account record, a credit on one account paired with a debit on another is the tick of the forward register F; that is a reading, and it decides nothing about which register is the clock.

He also repeats a claim that measurement rejects: that "the commonly accepted law that like charges repel and opposite charges attract is an inconsistent and flagrant error" [2].

Read beside the premise of Section 6.4, three of Russell's statements fit it. These are readings; they decide nothing.

- **Resistance.** "I believe that each of these forces resists the action of the other and in so doing gives to the two opposing sex forces the repeative effect which makes this universe of motion a continuous cycle" [2]. Under the one-unit bound, a held unit resists a second: the comparison is strict, and the repeated return is what makes the cycle.
- **No resting place.** "A condition or position of stability is impossible to maintain in this universe of changing potential between any two masses" [2]. A posting is made at every tick, so an account at zero must open; the tie between the two sides is broken by the orientation, not by rest.
- **Four units.** "Four units = one octave" [1, octave chart following p. 40], with one unit to each account, is the one-unit bound on four accounts.

9.11 Why we mined his structures and left his numbers

The counts make the method's case. Where Russell gives a number from his own laws, measurement usually rejects it: 216 of the 467 measurable claims disagree, and most of the failures are his laws carried past the data of his day. Where he gives an organizing structure, it matches RS in eleven of sixteen principles. And where RS has no statement at all, at the mover, his structure supplied conditions that logic could test. The test is what made them worth having. Of his fifteen statements about what moves, nine stand after checking, and the one that led to the breath was first refuted and then repaired by his own octave. The same method carried past the breath itself: the one-unit candidate, an addition to his text that first excluded the breath, became its premise once the accounts were four and in turn. The theorems of this paper do not depend on Russell being right. Theorem 4 holds from its two conditions, whoever stated them first. What Russell gave was the right two conditions to try.

10 Conclusion

RS's premises never pick the next posting, and equally cheap candidates differ only by a renaming and an orientation. Two conditions on a window of eight postings, balance and steady build-up, pick it. They admit one window up to orientation, and on four accounts with equal units that window is one act: four debits in turn, then four credits in the same order, each posting changing the account held longest. Its period meets the cube's only in three dimensions, and read through three bits it walks every corner of the cube. As a law of content it selects where the premises select nothing, permits loans, and on the postings leaves free exactly the renaming and the orientation.

On RS's own order of accounts, four in turn, one premise forces the two conditions: no account ever holds more than one unit. Given one posting per tick, that bound is the same statement as RS's own side rule, each posting taking the side of least cost on the posted account's balance; RS has that reading at model status and leaves the side at an empty account open, which is the orientation. With a fresh opening read as a plus, the bound gives the breath in every window, and nothing else does. It returns every unit four postings later, whether or not anything completes. It picks the side, not the account, and it carries its weight through two features, a signed balance and a posting at every tick. Read on the whole ledger instead of the posted account, the same rule gives the two-tick pair law. So the open questions of the act have narrowed to two. The next step is Open proposition P0: whether the ledger's accounts ever hold more than one unit. Beside it stands the tick (Open proposition P5): what one step of the ledger is, which register is its clock, and whether it posts four accounts in turn.

References

[1] W. Russell, *The Universal One*, Volume One: First Principles (Walter Russell, New York, 1926). Cited by the page of the 1926 edition; "law *n*" is law *n* of the volume's closing list of new laws and principles.

[2] W. Russell, *The Russell Genero-Radiative Concept, or The Cyclic Theory of Continuous Motion* (1930). Text-only edition of the University of Science and Philosophy; cited without page numbers.

[3] J. Washburn and M. Lj. Zlatanović, "Uniqueness of the canonical reciprocal cost," *Mathematics* **14**(6), 935 (2026). doi:10.3390/math14060935.

[4] S. Pardo-Guerra, J. Washburn and E. Allahyarov, "A coherence theorem for conserved comparison ledgers: Structural axioms pin down the scale, the cost, and the ratio," *Mathematics* **14**, 2672 (2026). doi:10.3390/math14152672.

[5] J. Washburn and M. Lj. Zlatanović, "The δ-calculus: from distinction to arithmetic," *Advances in Pure Mathematics* **16**(9), 705–739 (2026). doi:10.4236/apm.2026.169035.

A Census statistics

Part 10 includes Russell's closing list of 199 laws. The "Mind" and "Life, God and appearance" registers contribute only five measurable claims between them; Table 3 combines them.

Table 6. The census of the 1926 book.

Recorded claims, by part of the book		Distinct claims, by register	
Part 1	293	Mind	363
Part 2	340	Life, God and appearance	420
Part 3	539	Light	177
Part 4	450	Electricity and magnetism	460
Part 5	282	Motion and the wave	416
Part 6	389	Space, number and time	424
Part 7	499	Gravitation and radiation	336
Part 8	695	The universe and the solar system	412
Part 9	527	Energy and heat	279
Part 10	481	The elements	488
		Chemistry, colour and tone	272
Total	4,495	Total	4,047

Quotation check of the recorded claims	
exact match	2,163
close match (scan errors)	1,986
partial match	172
not found (mostly chart plates)	136
not checked	38
Glossary terms; numbered lists; numbers	292; 50; 893
Laws in Russell's closing list	199
Measurable claims (Table 3)	467

B The sixteen principles in Russell's words

	Principle	Russell's words
P1	The One substance	"There is but One universe. One Mind, One force, One substance" (Prelude).
P2	Thinking, the One force	"The force called 'thinking' which impels Mind into concentration and decentration in sequence is the only energy of the universe" (p. 14).
P3	Inertia	"Concept precedes motion. Motion begins in concept. Concept takes place in inertial equilibrium" (p. 102).
P4	Sex: the desire to divide and to unite	"Sex is the active desire of Mind for division into opposites, and its reactive desire for unity" (p. 23).
P5	The pulse	"All thinking is expressed in measurable and opposing impulses of opposing motion. All motion is action and reaction" (p. 15).

	Principle	Russell's words
P6	Balance	"there is no effect of motion in which its opposite effect is not simultaneously present and sequentially preponderant" (p. 115).
P7	Registration and appearance	"Thinking is the cause of motion and the periodicities, or states of motion, caused by thinking Mind are registered in light which man calls matter" (Prelude).
P8	Reproduction	"Reproduction means the repetition of a state of motion. Reproduction is repetition" (p. 27); "Light does not travel. It reproduces itself" (p. 30).
P9	Universal One-ness	"Everything that is, is of everything else that is. Nothing is of itself alone" (p. 40).
P10	Finitude	"This is a boundless, eternal, dimensionless universe of definite limitations both as to all cause and to all effects of cause" (p. 2).
P11	The constant of energy	"The universal constant of energy is never changing. It but appears to change. It is never unbalanced, never out of equilibrium" (p. 115).
P12	The wave	"All energy expresses itself in motion. All motion is expressed in waves" (p. 166).
P13	The octave	"One charging tone, and its exact mate in a discharging tone, balance as one unit constant of energy. There are four exactly equal unit constants of energy in each octave" (p. 82).
P14	The ten-octave cycle	"Ten octaves constitute one cycle" (p. 82).
P15	The universal ratios	"The seven tones of the universal constant are consecutively removed, one from the other, the square of the distance to the next highest potential" (law 125, p. 162).
P16	Potential position	"There is a true position for every potential" (law 94, p. 144).

Quotations follow the scanned 1926 text, with the scan's character errors corrected.